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Journal of Combinatorial Theory, Series B 93 (2005) 73–93

Journal of Combinatorial Theory Series B

www.elsevier.com/locate/jctb

2-Arc-transitive regular covers of complete graphs having the covering transformation group Z_p^3

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Received 24 March 2000

Available online 5 October 2004

Abstract

A family of 2-arc-transitive regular covers of a complete graph is investigated. In this paper, we classify all such covering graphs satisfying the following two properties: (1) the covering transformation group is isomorphic to the elementary abelian p-group Z_p^3, and (2) the group of fiber-preserving automorphisms acts 2-arc-transitively. As a result, new infinite families of 2-arc-transitive graphs are constructed.

MSC: 05C25; 20B25; 05E30

Keywords: Arc-transitive graph; Covering graph; Lifting; 2-Transitive group; Linear group

1. Introduction

Throughout this paper, a graph is assumed to be finite, simple and undirected. For the group- and graph-theoretic terminology, we refer the reader to [17,20]. For a graph X, every edge of X gives rise to a pair of opposite arcs. By $V(X)$, $E(X)$, $A(X)$ and Aut X, we denote

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0095-8956/$ - see front matter
doi:10.1016/j.jctb.2003.09.003

the vertex set, edge set, arc set and the full automorphism group of a graph X, respectively. The complete graph of order n is denoted by K_n. For any $v \in V(X)$, we use $N(v)$ to denote the neighborhood of v in X. A 2-*arc* of X is a sequence (v_0, v_1, v_2) of three vertices such that (v_0, v_1), $(v_1, v_2) \in A(X)$ but $v_0 \neq v_2$. The graph X is said to be 2-*arc-transitive* if Aut X acts transitively on the set of 2-arcs of X.

A graph X is called a *covering* (or *cover*) of a graph Y with the projection $p : X \to Y$ if there is a surjection $p : V(X) \to V(Y)$ such that $p|_{N(x)} : N(x) \to N(y)$ is a bijection for any vertex $y \in V(Y)$ and $x \in p^{-1}(y)$. The graph X is called the *covering graph* and Y is the *base graph*. A covering p is n-*fold* if $|p^{-1}(y)| = n$ for each $y \in V(Y)$. The *fiber* of an edge or a vertex is its preimage under p. An automorphism of X which maps a fiber to a fiber is said to be *fiber-preserving*. The group K of all automorphisms of X which fix each of the fibers setwise is called the *covering transformation group*. A cover X of Y is said to be *regular* (simply, L-*covering*) if there is a subgroup L of K acting freely and transitively (i.e., regularly) on each fiber. Moreover, if X is connected, then $L = K$.

This paper focuses on a classification of finite 2-arc-transitive graphs. The starting point for classifying such graphs is the following result of Praeger.

Proposition 1.1 (*Praeger [29, Theorem 4.1]*). *Let X be a connected graph and let G be a group of automorphisms of X which acts 2-arc-transitively on X. Assume that G has a normal subgroup N which has more than two orbits on $V(X)$. Let $\overline{X} = X/N$ denote the quotient graph of X by the N-action. Then, $\overline{X}$ is connected and N is semiregular on $V(X)$. Moreover, X is a regular cover of $\overline{X}$ with the covering transformation group N and G/N acts 2-arc-transitively on the quotient graph $\overline{X}$.*

By Proposition 1.1, the class of finite connected 2-arc-transitive graphs can be divided into two subclasses as follows:

(1) The 2-arc-transitive graphs X with the property that either (i) every nontrivial normal subgroup of Aut X acts transitively on $V(X)$, or (ii) every nontrivial normal subgroup of Aut X has at most two orbits on $V(X)$ and at least one of normal subgroups of Aut X has exactly two orbits on $V(X)$.
(2) The 2-arc-transitive regular covers of a graph given in (1).

A transitive permutation group G on a set Ω is said to be *primitive* if the property that for a subset $B \subset \Omega$, either $B^g = B$ or $B^g \cap B = \emptyset$ for each $g \in G$ is possible for only $B = \emptyset$, 1-element subset or the whole set Ω. It is *quasiprimitive* if every nontrivial normal subgroup of G is transitive on Ω. In particular, all primitive groups are quasiprimitive. However, there exist some quasiprimitive groups which are not primitive (see [29]). A structure theorem for finite quasiprimitive permutation groups was given in [29]. With these terminologies, in the first case (i) of the subclass (1), Aut X acts quasiprimitively on $V(X)$. During the last decades, many people have studied primitive or quasiprimitive finite 2-arc-transitive graphs (see [21,6–11] and others). In the second case (ii) of the subclass (1), X must be a bipartite graph and a reduction theorem for this case was given in [30]. In this paper, we examine only the second subclass (2).

Since the complete graph K_n is a standard example of a primitive 2-arc-transitive graph, it is interesting to investigate the *regular covers of K_n, when the group of fiber-preserving* *automorphisms acts* 2*-arc-transitively*. However, it seems infeasible to determine all of such covers. Therefore, we first need to determine the *minimal* such covers, that is, X cannot be such a cover of a graph which is also such a cover of K_n. Let X be such a cover with the covering transformation group K. It is easy to see that if K contains a nontrivial proper characteristic subgroup H, then X is a cover of a graph X_1 with the covering transformation group isomorphic to K/H, where X_1 is also a cover of K_n with the covering transformation group isomorphic to H. Therefore, in order to get a minimal regular cover, one may assume that K is a characteristically simple group. It is known [20, Theorem 9.12] that every characteristically simple group is either an elementary abelian group $\mathbb{Z}_p^n$ or a nonabelian one which is a direct product of isomorphic nonabelian simple groups. In this paper, we consider only the former abelian case. If K is either cyclic or $\mathbb{Z}_p^2$, it has been proved in [5] that K should be isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_2^2$, which induces a unique 2-fold cover and two families of 4-fold covers, respectively. In this paper, we classify all such covering graphs whose covering transformation groups K are isomorphic to $\mathbb{Z}_p^3$ and the groups of fiber-preserving automorphisms act 2-arc-transitively. As a by-product, we have new infinite families of 2-arc-transitive graphs. It is also worth mentioning that there exist some regular covering graphs X such that Aut X acts 2-arc-transitively but the group of fiber-preserving automorphisms cannot act 2-arc-transitively.

In the proof of the main theorem in [5], there is the following advantage: In the case when the covering transformation group K is isomorphic to either cyclic or $\mathbb{Z}_p^2$, the group K lies in the center of a subgroup G of Aut X, where G/K is isomorphic to either a nonabelian simple group or a 3-transitive subgroup of the affine transformation group $\mathrm{AGL}(m, 2)$. Under this condition, a result concerning a lifting of an automorphism of a base graph (see Proposition 2.2 below) can be applied. However, in the case of $K \cong \mathbb{Z}_p^3$, we do not have such advantage; this will cause new difficulties and hence we need a new method to handle them. This paper relies on the classification of 2-transitive groups and the structure of subgroups of the 3-dimensional general linear group $\mathrm{GL}(3, p)$, while the former relies on the classification of finite simple groups. All of the 2-arc-transitive regular covers found in this paper are presented as graphs derived from voltage assignments. To construct a 2-arc-transitive derived graph, the following several methods are used in this paper. In Section 3.1, we show how to use a lifting theorem (see Proposition 2.1) to determine whether an automorphism of the base graph can be lifted, even though the group K is not contained in the center of a subgroup G of Aut X as mentioned before. Aroused by this result, the authors obtained [4] a theoretical method (algorithm) for treating the case of $K = \mathbb{Z}_p^n$. In Section 3.2, a representation of the projective general linear group $\mathrm{PGL}(2, p)$ in $\mathrm{GL}(3, p)$ and a method of coset graphs are used; and in Section 3.3, some methods and results on permutation modules are used (see [21,27]). All of these ideas and methods might be applied for treating the case of $K = \mathbb{Z}_p^n$ for some small values n. However, to the best of the author's knowledge, it seems infeasible to get a complete classification for all n. Its difficulties come from the facts that so far the (reducible and irreducible) modular representations of 2-transitive groups over $GF(q)$ have not been completely classified and that neither has the subgroup structure of $\mathrm{GL}(n, p)$ been known except for some small values of n. The main result of this paper is Theorem 1.3 below.

A purely combinatorial description of a covering was introduced through a voltage graph by Gross and Tucker [17,18]. Let Y be a graph and K a finite group. A *voltage assignment* (or, *K-voltage assignment*) of the graph Y is a function $f : A(Y) \to K$ with the property that $f(u,v) = f(v,u)^{-1}$ for each $(u,v) \in A(Y)$. For convenience, we denote $f(u,v)$ by $f_{u,v}$. The values of f are called *voltages*, and K is called the *voltage group*. The *derived graph $Y \times_f K$ from a voltage assignment f* has as its vertex set $V(Y) \times K$ and as its edge set $E(Y) \times K$, so that an edge (e, g) of $Y \times_f K$ joins a vertex (u, g) to $(v, f_{u,v}g)$ for $(u, v) \in A(Y)$ and $g \in K$, where $e = uv$. Clearly, the graph $Y \times_f K$ is a covering of the graph Y with the first coordinate projection $p : Y \times_f K \to Y$, which is called the *natural projection*. For each $u \in V(Y)$, $\{(u, g) \mid g \in K\}$ is a fiber of u. Moreover, by defining $(u, g')^g := (u, g'g)$ for any $g \in K$ and $(u, g') \in V(Y \times_f K)$, K can be identified with a subgroup of Aut $(Y \times_f K)$ fixing each fiber setwise and acting regularly on each fiber. Therefore, p can be viewed as a *K-covering*. Conversely, each connected regular cover X of Y with the covering transformation group K can be described by a derived graph $Y \times_f K$ from some voltage assignment f. Given a spanning tree T of the graph Y, a voltage assignment f is said to be *T-reduced* if the voltages on the tree arcs are the identity. Gross and Tucker [18] showed that every regular cover X of a graph Y can be derived from a T-reduced voltage assignment f with respect to an arbitrary fixed spanning tree T of Y. Moreover, the voltage assignment f naturally extends to walks in Y. For any walk W of Y, let f_W denote the voltage of W. Finally, we say that an automorphism α of Y *lifts* to an automorphism $\overline{\alpha}$ of X if $\alpha p = p\overline{\alpha}$, where p is the covering projection from X to Y.

Before stating the main theorem, we first introduce four families of derived graphs whose voltage groups are elementary abelian p-groups.

Example 1.2. (1) Let Y be a connected graph with the betti number $\beta = \beta(Y) = |E(Y)| - |V(Y)|+1$. With a fixed spanning tree Y_1 of Y, we define the cover $Y \times_f \mathbb{Z}_p^\beta$ as follows: $f_{i,j} = 0$ for $(i, j) \in A(Y_1)$ and $\{f_{i,j} \mid (i, j) \in A(Y) \setminus A(Y_1)\}$ generates $\mathbb{Z}_p^\beta$, i.e., $\langle f_{i,j} \mid (i, j) \in A(Y) \setminus A(Y_1)\rangle = \mathbb{Z}_p^\beta$.

Clearly, each cover in this family is connected and it is *homological*. In particular, by $X_1(p)$, we denote such a cover $K_4 \times_f \mathbb{Z}_p^3$ for $Y = K_4$ and $\beta(K_4) = 3$.

(2) Let $V(K_5) = \{0, 1, 2, 3, 4\}$ and let K be the additive group of the 3-dimensional vector space $V(3, q)$ over the finite field $GF(q)$, where $q = p^k$ for a prime p and either $q = 5$ or $q = \pm 1 (\mathrm{mod}\ 10)$. By $X_2(q)$, we denote the cover $K_5 \times_f K$ defined as follows: $f_{0,j} = (0, 0, 0)$ for $1 \leqslant j \leqslant 4$, $f_{1,2} = (1, 0, 0)$, $f_{1,3} = (0, 1, 0)$, $f_{2,3} = (0, 0, 1)$, $f_{1,4} = (a, b, c)$, $f_{2,4} = (-b, -c, a)$ and $f_{3,4} = (c, -a, -b)$, where $a = \frac{1+\sqrt{5}}{4}$, $b = \frac{1-\sqrt{5}}{4}$ and $c = \frac{\sqrt{5}}{2}$.

Let H be the subgroup of K generated by all the voltages on cotree-arcs. Then, it is easy to see that $|H| = p^3$ (resp. p^6) if $\sqrt{5} \in GF(p)$ (resp. $\sqrt{5} \in GF(q) \setminus GF(p)$). Therefore, in the case of $q = p$, the cover $X_2(p)$ is connected; and in the case of $q = p^2$ and $\sqrt{5} \in GF(q) \setminus GF(p)$, the cover $X_2(p^2)$ is isomorphic to the cover $K_5 \times_f \mathbb{Z}_p^6$ defined in (1). Moreover, the cover $X_2(q)$ is a union of connected covers each of which is isomorphic to either $X_2(p)$ or $K_5 \times_f \mathbb{Z}_p^6$ defined in (1).

(3) Let $q = p^k$ be a prime power with a prime p, and let us identify $V(K_{1+q})$ with the projective line $PG(1, q) = GF(q) \cup \{\infty\}$. Let K be the additive group of the vector space

$V(3,q)$. Denote by $X_3(q)$ the cover $K_{1+q} \times_f K$ defined as follows: $f_{\infty,j} = (0, 1, 2j)$ and $f_{i,j} = \left(\frac{1}{i-j}, \frac{i+j}{i-j}, \frac{2ij}{i-j}\right)$ for all $i \neq j$ in $GF(q)$.

Let H be the subgroup of K generated by the voltages on closed walks of K_{1+q}. Then, the cover $X_3(q)$ is a union of $|K : H|$ isomorphic connected covers. In particular, if $q = p \geqslant 5$, then $H = K \cong \mathbb{Z}_p^3$ (see Section 3.2), and in this case the cover $X_3(p)$ is connected.

(4) Let Ω be a finite set. Given a $\Delta \in P(\Omega)$, the power set of Ω, let χ_Δ denote the characteristic function of Δ, that is, if $\chi_\Delta(i) = 1$ for $i \in \Delta$ and $\chi_\Delta(i) = 0$ for $i \notin \Delta$. Then, the set $V(\Omega)$ of all characteristic functions χ_Δ, where $\Delta \in P(\Omega)$, forms a vector space over $GF(2)$ with the rule: $(a\chi_\Delta + b\chi_\Gamma)(i) = a\chi_\Delta(i) + b\chi_\Gamma(i)$ for any $a, b \in GF(2)$ and $\chi_\Delta, \chi_\Gamma \in V(\Omega)$. Clearly, a natural basis for $V(\Omega)$ is the set of characteristic functions $\chi_{\{i\}}$ for all $i \in \Omega$.

Now, for $m \geqslant 3$, we identify $V(K_{2^m})$ with $V(m, 2)$. Let $\Omega = PG(m-1, 2)$ be the $(m-1)$-dimensional projective space over the field $GF(2)$, while we identify Ω with $V(m,2) \setminus \{0\}$. Then, $V = V(\Omega)$ is a $(2^m - 1)$-dimensional vector space. Note that a 1-dimensional subspace of $PG(m-1,2)$ can be written as $\{i, j, i+j\}$ for all $i \neq j$ in Ω, while a 2-dimensional subspaces of $PG(m-1, 2)$ can be written as $\{i, j, k, i+j, j+k, k+i\}$ for any three distinct elements i, j, k in Ω. Let V_1 and V_2 be the subspaces of $V(\Omega)$ generated by the characteristic functions of all 1-dimensional subspaces and of all 2-dimensional subspaces of $PG(m-1, 2)$, respectively. Then, the dimension of the quotient space V_1/V_2 is $\frac{m(m-1)}{2}$ (cf. Lemma 3.8). Let K be the corresponding additive group of V_1/V_2. Now, for $m \geqslant 3$, we define the cover $X_4(m) = K_{2^m} \times_f K$ as follows: $f_{0,j} = \overline{0} := V_2$ and $f_{i,j} = \overline{\chi}_{\{i,j,i+j\}} := \chi_{\{i,j,i+j\}} + V_2$ for all $i \neq j$ in Ω.

Clearly, the cover $X_4(m)$ is connected. Let $\mathbf{1} = \chi_\Omega$ be the constant characteristic function on Ω. If $m = 3$, then $V_2 = \langle \mathbf{1} \rangle$, and $V_1 = V_2 \oplus W$, where W is the unique 3-dimensional subspace of V generated by all the functions of the complements of lines in $PG(m-1, 2)$ (cf. [21, Theorem 5.1]).

Now, we state the main result of this paper, which will be proved in Section 3.

Theorem 1.3. *Let X be a connected regular cover of a complete graph K_n $(n \geqslant 4)$ satisfying the following two properties*: (1) *the covering transformation group is isomorphic to the elementary abelian p-group $\mathbb{Z}_p^3$, and* (2) *the group of fiber-preserving automorphisms acts* 2*-arc-transitively. Then, X is isomorphic to one of the graphs belonging to the four families listed in Example* 1.2: $X_1(p) = K_4 \times_f \mathbb{Z}_p^3$, $X_2(p) = K_5 \times_f \mathbb{Z}_p^3$ for $p = 5$ *or* $p \equiv \pm 1 (\mathrm{mod}\ 10)$, $X_3(p) = K_{1+p} \times_f \mathbb{Z}_p^3$ *for* $p \geqslant 5$, *and* $X_4(3) = K_8 \times_f \mathbb{Z}_2^3$. *Conversely, for each cover contained in the four families in Example* 1.2, *the group of fiber-preserving automorphisms acts* 2*-arc-transitively*.

In [28], Praeger suggested a possibility of forming minimal normal covers. It will be proved in Proposition 3.10 that all the p^3-fold covers defined in $X_i(p)$ where i=1, 2, 3, and $X_4(3)$ are minimal regular covers of a complete graph. It means that any of such covers is not a cover of a graph which is a cover of a complete graph having the covering transformation group either $\mathbb{Z}_p$ or $\mathbb{Z}_p^2$.

Finally, we would like to mention some other results which also deal with the covers of a complete graph. In [14], Godsil et al. classified the distance-transitive antipodal covers of a complete graphs K_n, depending on the classification of finite 2-transitive permutation groups. All such covers with the diameter $d \geqslant 3$ are 2-arc-transitive with the cyclic covering transformation groups. In [15], Gardiner and Praeger investigated the arc-transitive covers of a complete graph K_n so that the stabilizer of a block (fiber) B acts 2-transitively on B, and they proved that there is a unique exceptional graph for which the size of fibers is greater than n. All of the covers in [14] or [15] are not necessarily regular. Moreover, a direct comparison shows that none of the covers listed in Theorem 1.3 was mentioned in [14] or [15]. In [25], the covers of K_n are characterized for which the whole automorphism group S_n lifts. For other results dealing with the covers of a complete graph, we refer the reader to [12,13,16,26].

2. Preliminaries

We start by introducing some preliminary results which will be used in Section 3. The first two propositions are related to a lifting criterion of an automorphism of a base graph with respect to a voltage assignment.

Proposition 2.1 (*Malnič [24, Corollary 4.3]*). *Let Y be a connected graph and let X be a cover of Y derived from a voltage assignment f. Then, an automorphism α of Y can be lifted to an automorphism of X if and only if, for each closed walk W in Y, we have $f_{W^\alpha} = 1$ if and only if $f_W = 1$.*

Proposition 2.2. *Let K be a finite group, and let $X = Y \times_f K$ be a connected regular cover of a graph Y derived from a voltage assignment f with the voltage group K. If $\alpha \in \mathrm{Aut}\, Y$ is an automorphism one of whose liftings $\tilde{\alpha}$ centralizes K, considered as the covering transformation group, then for any closed walk W in Y, there exists $k \in K$ such that $f_{W^\alpha} = k f_W k^{-1}$. In particular, if K is abelian, $f_{W^\alpha} = f_W$ for any closed W of Y.*

Proof. Suppose that K and α satisfy the condition of the proposition. By (u, k), we denote a vertex of $V(X)$ for any $u \in V(Y)$ and $k \in K$. Then, we have $(u, k)^{k'} = (u, kk')$ for any $k' \in K$. Take any closed walk

$$W = u_0, u_1, \ldots, u_{n-1}, u_n (= u_0)$$

in Y and consider a lifted walk

$$\tilde{W} = (u_0, 1), (u_1, f_{u_0,u_1}), \ldots, (u_n, f_W)(= (u_0, f_W)),$$

in X. Then, one can get

$$\tilde{W}^{\tilde{\alpha}} = (u_0, 1)^{\tilde{\alpha}}, (u_1, f_{u_0,u_1})^{\tilde{\alpha}}, \ldots, (u_0, f_W)^{\tilde{\alpha}}.$$

Let $(u_0, 1)^{\tilde{\alpha}} = (u_0^{\alpha}, k)$. Since $\tilde{\alpha}$ centralizes K, we have

$$(u_0, f_W)^{\tilde{a}} = (u_0, 1)^{f_W \tilde{\alpha}} = ((u_0, 1)^{\tilde{\alpha}})^{f_W} = (u_0^{\alpha}, k)^{f_W} = (u_0^{\alpha}, kf_W) = (u_0^{\alpha}, (kf_W k^{-1})k).$$

It shows that the lifted walk $\tilde{W}^{\tilde{a}}$ of W^{α} starts from (u_0^{α}, k) and ends at $(u_0^{\alpha}, (kf_W k^{-1})k)$. Hence, $f_{W^{\alpha}} = kf_W k^{-1}$ by the uniqueness of a path-lifting. $\square$

In Proposition 2.2, if K is not abelian, one cannot conclude that $f_{W^{\alpha}} = f_W$ for any closed walk W of Y. The following counter-example was suggested by a referee.

Example 2.3. Let $V(K_4) = \{w, x, y, z\}$ and let S_3 be the symmetric group on three elements $\{1, 2, 3\}$. We define a cover $X = K_4 \times_f S_3$ as follows: $f_{w,x} = f_{w,y} = f_{w,z} = 1$, $f_{x,y} = (12)$, $f_{y,z} = (23)$ and $f_{z,x} = (31)$. Take $\alpha = (wx)(yz) \in \operatorname{Aut} K_4 \cong S_4$. Define a map $\tilde{\alpha}$ on the covering graph X as follows: For any $g \in S_3$, let $(w, g)^{\tilde{\alpha}} = (x, (23)g)$, $(x, g)^{\tilde{\alpha}} = (w, (23)g)$, $(y, g)^{\tilde{\alpha}} = (z, (123)g)$ and $(z, g)^{\tilde{\alpha}} = (y, (132)g)$. Then, $\tilde{\alpha} \in \operatorname{Aut} X$ is a lifting of α and it centralizes S_3, considered as the covering transformation group. But, for a closed walk $W = w, x, y, z, w$ in K_4, $W^{\alpha} = x, w, z, y, x$ and $f_W = (12)(23) \neq (23)(12) = f_{W^{\alpha}}$.

In addition, Proposition 2.2 also gives a correction of [5, Lemma 2.2], in which K is not assumed to be abelian. However, since the group K considered in [5] is either cyclic or $\mathbb{Z}_p^2$, all the arguments and results in [5] are still valid.

A proof of the following lemma is not difficult and it can be found in [4].

Lemma 2.4. *Let Y be a connected graph and let X be a connected regular cover of Y with the covering transformation group $K = \mathbb{Z}_p^n$ for a prime p. Then, $n \leqslant \beta(Y)$, the betti number of Y. In particular, if $n = \beta(Y)$, then X is unique up to isomorphism and any automorphism of Y can be lifted to an automorphism of X.*

The next proposition is an extraction from a complete list of doubly transitive groups, given by Cameron and Kantor in [2,3, Corollary 8.3]).

Proposition 2.5. *Let G be a* 3*-transitive permutation group of degree at least* 5. *Then, one of the following cases occurs.*

(1) *The socle of G is* 3*-transitive*;
(2) $\mathrm{PSL}(2, q) \leqslant G \leqslant \operatorname{Aut} \mathrm{PSL}(2, q)$, *where the projective special linear group* $\mathrm{PSL}(2, q)$ *is the socle* $\mathrm{soc}(G)$ *of G which does not act* 3*-transitively, and G acts on the projective geometry $PG(1, q)$ in a natural way, having degree $q + 1$, with $q \geqslant 5$ an odd prime power. Write* $\mathrm{P\Gamma L}(2, q) = \operatorname{Aut} \mathrm{PSL}(2, q)$;
(3) $G = \mathrm{AGL}(m, 2)$, *an affine group with $m \geqslant 3$*;
(4) $G = \mathbb{Z}_2^4 \rtimes A_7 < \mathrm{AGL}(4, 2)$.

The following result may be deduced from the Bloom's determination of the subgroups of $\mathrm{PSL}(3, q)$ in [1].

Proposition 2.6. *Let $G = \mathrm{GL}(3, p)$ for an odd prime p. Then,*

(1) *any nontrivial subgroup H of G which does not contain an elementary abelian normal subgroup of order $\geqslant 2$ is isomorphic to one of the following groups*:
 (i) $\mathrm{PSL}(2, 5)$ *with* $p \equiv \pm 1 (\mathrm{mod}\ 10)$;
 (ii) $\mathrm{PSL}(2, 7)$ *with* $p^3 \equiv 1 (\mathrm{mod}\ 7)$;
 (iii) $\mathrm{PSL}(2, p)$ *or* $\mathrm{PGL}(2, p)$ *for* $p \geqslant 5$.
 Moreover, G has exactly one conjugacy class of subgroups isomorphic to each subgroup H listed in (i)-(iii).

(2) *G contains neither the affine group $\mathrm{AGL}(m, 2)$ for $m \geqslant 3$ nor $\mathbb{Z}_2^4 \rtimes A_7$.*

Lemma 2.7. *Let p be an odd prime and $p^3 \equiv 1 (\mathrm{mod}\ 7)$. Then, as a subgroup of* $\mathrm{GL}(3, p)$, $\mathrm{PSL}(2, 7)$ *has no orbits of length* 7 *in its action on the space $V(3, p)$.*

Proof. By Proposition 2.6, $\mathrm{GL}(3, p)$ has only one conjugacy class of subgroups isomorphic to $\mathrm{PSL}(2, 7)$. So, it is enough to prove the lemma for any representative in the conjugacy class. Now, we take a subgroup $T \cong \mathrm{PSL}(2, 7)$ in $\mathrm{GL}(3, p)$ as follows: Let

$$a_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad b_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

be two matrices in the special linear group $\mathrm{SL}(2, 7)$, and let

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} r & \frac{1}{2} & -\frac{1}{2} \\ r & -\frac{1}{2} & \frac{1}{2} \\ 0 & r + \frac{1}{2} & r + \frac{1}{2} \end{pmatrix}$$

be two matrices in $\mathrm{GL}(3, p)$, where r satisfies $r^2 + 2r + 1 = 0$. We have $\langle a_1, b_1 \rangle = \mathrm{SL}(2, 7)$ and the mapping $\phi : a_1 \mapsto a,\ \ b_1 \mapsto b$ is a homomorphism from $\mathrm{SL}(2, 7)$ into $\mathrm{GL}(3, p)$. Moreover, the image T of ϕ is a subgroup isomorphic to $\mathrm{PSL}(2, 7)$ (see [1, Lemma 6.5]).

Suppose that the subgroup T have an orbit of length 7 in its action on the space $V(3, p)$. Then, for each of such orbits, every involution in T must fix a point in the orbit and every Sylow 7-subgroup of T must be transitive on the orbit. (Note that $\langle b \rangle$ is a Sylow 7-subgroup of T.) Take any vector $\mathbf{v} = (x, y, z) \in V(3, p)$. Since $\mathbf{v}a = (x, -y, -z)$, the fixed points of a are of the form $(x, 0, 0)$. Now, the $\langle b \rangle$-orbit containing $(x, 0, 0)$ is: $\{(x, 0, 0), (rx, \frac{x}{2}, -\frac{x}{2}), (-\frac{x}{2}, -\frac{x}{2}, -rx), -(rx, \frac{x}{2}, \frac{x}{2}), (rx + \frac{x}{2}, 0, rx + \frac{x}{2}), (\frac{x}{2}, xr, -\frac{x}{2}), (-rx - \frac{x}{2}, -rx - \frac{x}{2}, 0)\}$. However, for any $x \neq 0$, an easy check shows that a cannot preserve setwise this $\langle b \rangle$-orbit of length 7. So, T cannot have an orbit of length 7. □

Lemma 2.8. *Suppose $G = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$, where $a^4 = b^4 = c^4 = 1$. Let $K = \langle a^2, b^2, c^2 \rangle$ and let H be the subgroup of* $\mathrm{Aut}\, G$ *generated by all the elements fixing K pointwise. Then, $H \cong \mathbb{Z}_2^9$ and* $(\mathrm{Aut}\, G)/H \cong \mathrm{PSL}(2, 7)$. *Moreover, there exists no complement T of H in* $\mathrm{Aut}\, G$ *such that T has an orbit of length* 7 *in $G \setminus K$.*

Proof. Identifying G with the additive group of the 3-dimensional module over the ring $\mathbb{Z}_4$, $\mathrm{Aut}\,G$ is isomorphic to the multiplicative group of all 3×3 matrices over the ring $\mathbb{Z}_4$ with determinant ± 1. Since K is a characteristic subgroup of G, $\mathrm{Aut}\,G$ fixes K setwise. Consider the action of $\mathrm{Aut}\,G$ on K. Then, the kernel H of this action consists of all matrices of the diagonal entries ± 1 and the off-diagonal entries 0 or 2, and we have $(\mathrm{Aut}\,G)/H \cong \mathrm{PSL}(2,7)$. A direct computation shows that $H \cong \mathbb{Z}_2^9$.

Now, suppose that the kernel H has a complement T in $\mathrm{Aut}\,G$ having an orbit Δ of length 7 in $G \setminus K$. Take a vector $\mathbf{v} = (1, 0, 0)$. Without any loss of generality, we may assume that $\mathbf{v} \in \Delta$. (Otherwise, we may replace T by its conjugate subgroup, because $\mathrm{Aut}\,G$ is transitive on $G \setminus K$.) Let S be the subgroup of $\mathrm{Aut}\,G$ consisting of the matrices of determinant 1. Then, $|\mathrm{Aut}\,G : S| = 2$ and $T \leqslant S$, because T is simple. Let ε be a ring homomorphism from $\mathbb{Z}_4$ to $\mathbb{Z}_2$ with $1^\varepsilon = 1$. Then, ε induces a homomorphism ϕ from $\mathrm{Aut}\,G$ to $\mathrm{GL}(3,2)$ with kernel H. The restriction ϕ_T of ϕ on T is an isomorphism from T onto $\mathrm{GL}(3,2)$, which maps the elements in the stabilizer $T_{\mathbf{v}}$ to matrices in $\mathrm{GL}(3,2)$ having the first row $(1, 0, 0)$. Take an involution r in $\mathrm{GL}(3,2)$ as

$$r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then, the preimage $\phi_T^{-1}(r)$ of r is contained in the subgroup $T_{\mathbf{v}}$. So, we may assume that

$$\phi_T^{-1}(r) = \begin{pmatrix} 1 & 0 & 0 \\ a & c & e \\ b & d & f \end{pmatrix},$$

where e and d are ± 1, and a, b, c and f are even. Since $\phi_T^{-1}(r) \in S$ and $cf = 0$ in $\mathbb{Z}_4$, we have $de = -1$. However, since $r^2 = 1$ and ϕ_T is an isomorphism, we have $(\phi_T^{-1}(r))^2 = 1$, which implies that $de = 1$. This is a contradiction to complete the proof. $\square$

Finally, we recall the following terminology for a later use. Let G be a group and let $H \leqslant G$ with the trivial core, that is $\bigcap_{g \in G} g^{-1}Hg = 1$. Let $D = Hg_0H$ be a double coset for some $g_0 \in G \setminus H$. Then, one can construct a digraph $X(G, H, D)$, called a *coset digraph*, having the vertex set $\{Hg \mid g \in G\}$ and the arc set $\{(Hg_1, Hg_2) \mid g_2g_1^{-1} \in D\}$. Clearly, G acts transitively on both the vertex set and the arc set by the right multiplication. Furthermore, if $D = D^{-1}$, then $X(G, H, D)$ is undirected by identifying two arcs (Hg_1, Hg_2) and (Hg_2, Hg_1) with one undirected edge Hg_1Hg_2.

3. Proof of Theorem 1.3

To prove Theorem 1.3, let $X = K_n \times_f K$ be a connected regular cover of a complete graph K_n ($n \geqslant 4$) whose covering transformation group K is isomorphic to $\mathbb{Z}_p^3$ and whose fiber-preserving automorphism subgroup A acts 2-arc-transitively on X. Let $\mathcal{F}$ be the set of fibers. Then, A is the largest subgroup of $\mathrm{Aut}\,X$ having $\mathcal{F}$ as an imprimitive block system and K is the kernel of the action of A on $\mathcal{F}$. Since A acts 2-arc-transitively on X, so does A/K on the base graph K_n. This forces A/K to be a 3-transitive group on $V(K_n)$ and so it is one of groups listed in Proposition 2.5. Take a spanning tree Y_0 in K_n which is a star having the base vertex $0 := p(F)$ for a fixed fiber F, where p is the covering projection from X to K_n. We can assume that the voltage $f = \mathbf{0}$ on the arcs of the spanning tree Y_0. Assume $n = 4$. Then, K_n has exactly three cotree edges. Hence, X should be isomorphic to $X_1(p)$ by Lemma 2.4. In what follows, we assume $n \geqslant 5$.

Let $C_A(K)$ be the centralizer of K in A. Suppose that the subgroup $C_A(K)/K$ of A/K is 3-transitive on $V(K_n)$. Consider the action of $C_A(K)/K$ on the set of triangles $0, i, j$ of K_n, where we assume $f_{0,j} = \mathbf{0}$ for any $j \neq 0$. By Proposition 2.2, we know that for any triangle W in K_n, each element in $C_A(K)/K$ preserves the voltage of W. Therefore, from the 3-transitivity of $C_A(K)/K$ on $V(K_n)$ we know that all such triangles have the same voltage, and consequently, all the cotree arcs in K_n have the same voltage. Since X is assumed to be connected, all voltages assigned to the cotree arcs in K_n should generate K. It means that K is a cyclic group, a contradiction.

Next, suppose that $C_A(K)/K$ is isomorphic to an almost simple group T with $\mathrm{PSL}(2,q) \leqslant T \leqslant \mathrm{P\Gamma L}(2,q)$ for some odd prime power $q \geqslant 5$, which acts on $PG(1,q)$ in a natural way. Then, the point stabilizer of x has at most two orbits on the triples $\{x, y, z\}$ for $y, z \in PG(1,q) \setminus \{x\}$. We identify $V(K_n)$ with $PG(1,q)$ and choose x as the base vertex of the spanning tree. With an argument similar to the previous paragraph, one can get from Proposition 2.2 that all the voltages assigned to the cotree arcs have at most two different values. Therefore, they cannot generate the group $K \cong \mathbb{Z}_p^3$, which forces that X is disconnected, a contradiction again. Hence, one may assume from now on that $C_A(K)/K$ is neither 3-transitive nor almost simple with the socle $\mathrm{PSL}(2,q)$.

Since $A/C_A(K)$, which is a factor group of A/K, is isomorphic to a subgroup of $\mathrm{Aut}\, K \cong \mathrm{GL}(3,p)$, it follows from Propositions 2.5 and 2.6 that $C_A(K) = K$ when A/K is an almost simple 3-transitive group; and $C_A(K)/K \cong \mathbb{Z}_2^m$ with $m \geqslant 3$ when A/K is an affine 3-transitive group. Moreover, by a careful inspection of Propositions 2.5 and 2.6, one can find only the following three possibilities:

Case (1): $\mathrm{soc}(A/K) \cong \mathrm{PSL}(2,5) \cong A_5$, $p = 5$ or $p \equiv \pm 1 (\mathrm{mod}\ 10)$, and $n = 5$. This case will be treated in Section 3.1 to conclude that $X \cong X_2(p)$;

Case (2): $A/K \cong \mathrm{PGL}(2,p)$ for $p \geqslant 5$ and $n = 1 + p$. This case will be treated in Section 3.2 to conclude $X \cong X_3(p)$;

Case (3): $A/K \cong \mathrm{AGL}(3,2) \cong \mathbb{Z}_2^3 \rtimes \mathrm{GL}(3,2)$, $p^3 \equiv 1 (\mathrm{mod}\ 7)$ and $n = 8$, noting that p may be 2. This case will be treated in Section 3.3 to conclude $X \cong X_4(3)$.

3.1. *When* $\mathrm{soc}(A/K)$ *is isomorphic to* A_5

In this subsection, we suppose that $\mathrm{soc}(A/K) \cong A_5$ and $n = 5$, and that either $p = 5$ or $p \equiv \pm 1 (\mathrm{mod}\ 10)$. In this case, we divide our proof into the following two lemmas.

Lemma 3.1. *X must be isomorphic to $X_2(p)$.*

Proof. Let $X = K_5 \times_f K$ with a voltage assignment $f : A(K_5) \to K$ and let $V(K_5) = \{0, 1, 2, 3, 4\}$. We identify K with the additive group of the 3-dimensional vector space

$V(3, p)$ over $GF(p)$, so that the identity element in K is identified with the zero vector $\mathbf{0}$. Take a basis $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ in K. Take the star Y_0 with the base vertex 0 as a spanning tree of K_5. Moreover, one may assume that $f_{0,i} = \mathbf{0}$ for any $i \in V_1 := \{1, 2, 3, 4\}$.

It is easy to check that for the induced subgraph $K_5[V_1]$ of K_5, if the voltages assigned to the respective arcs in any triangle and in any claw are linearly dependent in $K = V(3, p)$, then the group generated by all voltages has order less than p^3. This contradicts the connectedness of X. So, we have the following two different cases:

(1) In $K_5[V_1]$, the three voltages on the respective arcs in any triangle are linearly dependent, but there exists a claw such that the three voltages on the arcs in this claw are linearly independent. Hereafter, for any three distinct vertices i, j, k in $V(K_5)$, we use $(i, j, k)^n$ to denote the walk $(i, j, k, i, j, k, \ldots, i, j, k)$, (repeat n times). Without any loss of generality, one may assume that $f_{1,2} = \mathbf{x},\ f_{1,3} = \mathbf{y},\ f_{1,4} = \mathbf{z}$ and $f_{2,3} = a\mathbf{x} + b\mathbf{y}$. Take a closed walk $W = ((0, 1, 2)^a, (0, 1, 3)^b, 0, 3, 2, 0)$. We have $f_W = af_{1,2} + bf_{1,3} - f_{2,3} + (a + b) f_{0,1} + (1 - a) f_{0,2} - (b - 1) f_{0,3} = \mathbf{0}$. Since A_5 lifts, $f_{W^\alpha} = \mathbf{0}$ for each $\alpha \in A_5$ by Proposition 2.1. Since $f_{W^{(243)}} = af_{1,4}+bf_{1,2}-f_{4,2}+(a+b) f_{0,1}+(1-a) f_{0,4}-(b-1) f_{0,2} = \mathbf{0}$, we have $f_{2,4} = -b\mathbf{x} - a\mathbf{z}$. Since $f_{W^{(12)(34)}} = \mathbf{0}$ and $f_{W^{(012)}} = \mathbf{0}$, we have

$$-a\mathbf{x} - \mathbf{z} + bf_{2,4} = \mathbf{0} \quad \text{and} \quad (a + b)\mathbf{x} + (1 - b)\mathbf{y} + bf_{2,3} = \mathbf{0}. \tag{3.1}$$

Substituting the values of $f_{2,3}$ and $f_{2,4}$ to (3.1), we get the following system of equations in $GF(p)$:

$$a + b^2 = 0,\ 1 + ab = 0 \quad \text{and} \quad a + b + ab = 0.$$

However, it is easy to check that this system has no solutions.

(2) In $K_5[V_1]$, there exists a triangle such that three voltages assigned to its arcs are linearly independent. Without any loss of generality, one may assume that $f_{1,2} = \mathbf{x},\ f_{1,3} = \mathbf{y}$, $f_{2,3} = \mathbf{z}$ and $f_{1,4} = a\mathbf{x}+b\mathbf{y}+c\mathbf{z}$. Take a closed walk $W = ((0, 1, 2)^a, (0, 1, 3)^b, (0, 2, 3)^c, 0, 4, 1, 0)$. Then, $f_W = af_{1,2} + bf_{1,3} + cf_{2,3} - f_{1,4} + (a + b - 1) f_{0,1} + (c - a) f_{0,2} - (b + c) f_{0,3} + f_{0,4} = \mathbf{0}$. Similar to (1), since $f_{W^{(132)}} = \mathbf{0}$ and $f_{W^{(123)}} = \mathbf{0}$, we have $f_{3,4} = c\mathbf{x} - a\mathbf{y} - b\mathbf{z}$ and $f_{2,4} = -b\mathbf{x} - c\mathbf{y} + a\mathbf{z}$. Since $f_{W^{(12)(34)}} = \mathbf{0}$ and $f_{W^{(02)(13)}} = \mathbf{0}$, we have

$$-a\mathbf{x}-\mathbf{z}+cf_{1,4}+bf_{2,4} = \mathbf{0} \quad \text{and} \quad (b+c)\mathbf{x}-b\mathbf{y}+(a+b-1)\mathbf{z}+ f_{2,4}- f_{3,4} = \mathbf{0}. \tag{3.2}$$

Substituting the values of $f_{1,4}$, $f_{2,4}$ and $f_{3,4}$ to (3.2), one can get the following system of equations in $GF(p)$:

$$-a + ac - b^2 = 0, \quad -1 + c^2 + ab = 0, \quad a - b - c = 0 \quad \text{and} \quad 2a + 2b - 1 = 0.$$

Solving this system of equations, one can get $4a^2 - 2a - 1 = 0, \quad b = \frac{1}{2} - a \quad$ and $\quad c = 2a - \frac{1}{2}$. However, the first equation has a solution if and only if $p = 5$ or $p \equiv \pm1 (\mathrm{mod}\ 10)$. If $p \equiv \pm1 (\mathrm{mod}\ 10)$, we have two solutions; $(a, b, c) = (\frac{1+\sqrt{5}}{4}, \frac{1-\sqrt{5}}{4}, \frac{\sqrt{5}}{2})$ and $(a, b, c) = (\frac{1-\sqrt{5}}{4}, \frac{1+\sqrt{5}}{4}, \frac{-\sqrt{5}}{2})$. The graph determined by the first solution for (a, b, c) is nothing but $X_2(p)$. It is easy to see that the graph determined by the second solution for (a, b, c) is isomorphic to $X_2(p)$ if we assume $f_{1,2} = \mathbf{y}$ and $f_{1,3} = \mathbf{x}$. If $p = 5$, we have $(a, b, c) = (-1, -1, 0)$. $\square$

In Lemma 3.1, we identified K with the additive group of the 3-dimensional vector space $V(3, p)$ over $GF(p)$. If the field $GF(p)$ is replaced by $GF(q)$ for $q = p^k$, one can get the family $X_2(q)$.

Lemma 3.2. *For each cover in $X_2(q)$, the group of fiber-preserving automorphisms acts 2-arc-transitively on the covering graph.*

Proof. Since A/K is isomorphic to either A_5 or S_5, it suffices to show that A_5 lifts. Since A_5 is generated by (13)(24) and (012), A_5 lifts if and only if these two generators lift.

Let W be a closed walk in K_5 with $f_W = \mathbf{0}$. One may assume that the arc (i, j) (resp. (j, i)) appears $\ell_{i,j}$ (resp. $\ell_{j,i}$) times in W and let $t_{i,j} = \ell_{i,j} - \ell_{j,i}$. Since $f_{i,j} = -f_{j,i}$, we get $t_{i,j} = -t_{j,i}$. Then, $f_W = \sum_{0\leqslant i<j\leqslant 4} t_{i,j} f_{i,j} = \mathbf{0}$. Substituting the values of $f_{i,j}$ to it, one can get the following three relations between $\{t_{i,j}\}$:

$$\begin{aligned} t_{1,2} &= -a\, t_{1,4} + b\, t_{2,4} - c\, t_{3,4},\\ t_{1,3} &= -b\, t_{1,4} + c\, t_{2,4} + a\, t_{3,4},\\ t_{2,3} &= -c\, t_{1,4} - a\, t_{2,4} + b\, t_{3,4}. \end{aligned} \tag{3.3}$$

Since W is a closed walk, the numbers of arcs in W coming from i and going into i are the same for any vertex i in $V(K_5)$. So, we get

$$\begin{aligned} t_{0,1} &= t_{1,2} + t_{1,3} + t_{1,4} = (1 - a - b)t_{1,4} + (b + c)t_{2,4} + (a - c)t_{3,4},\\ t_{0,2} &= t_{2,1} + t_{2,3} + t_{2,4} = (a - c)t_{1,4} + (1 - a - b)t_{2,4} + (c + b)t_{3,4},\\ t_{0,3} &= t_{3,1} + t_{3,2} + t_{3,4} = (b + c)t_{1,4} + (a - c)t_{2,4} + (1 - a - b)t_{3,4},\\ t_{0,4} &= t_{4,1} + t_{4,2} + t_{4,3} = -t_{1,4} - t_{2,4} - t_{3,4}. \end{aligned} \tag{3.4}$$

Let $\alpha = (13)(24)$. Then, we have

$$f_{W^\alpha} = \sum_{0\leqslant i<j\leqslant 4} t_{i,j} f_{i^\alpha, j^\alpha} = t_{1,2} f_{3,4} + t_{1,3} f_{3,1} + t_{2,3} f_{4,1} + t_{1,4} f_{3,2} + t_{2,4} f_{4,2} + t_{3,4} f_{1,2}.$$

Substituting the values of $f_{i,j}$ to it and by using (3.3), we get

$$\begin{aligned} f_{W^\alpha} &= (ct_{1,2} - at_{2,3} + bt_{2,4} + t_{3,4})\mathbf{x}\\ &\quad +(-at_{1,2} - t_{1,3} - bt_{2,3} + ct_{2,4})\mathbf{y}\\ &\quad +(-bt_{1,2} - ct_{2,3} - t_{1,4} - at_{2,4})\mathbf{z}\\ &= \big((bc + a^2 + b)t_{2,4} - (c^2 + ab - 1)t_{3,4}\big)\,\mathbf{x}\\ &\quad + \big((a^2 + b + bc)t_{1,4} + (ac - a - b^2)t_{3,4}\big)\,\mathbf{y}\\ &\quad + \big(-(1 - ab - c^2)t_{1,4} + (-b^2 + ac - a)t_{2,4}\big)\,\mathbf{z}. \end{aligned} \tag{3.5}$$

Since $(a, b, c) = (\frac{1+\sqrt{5}}{4}, \frac{1-\sqrt{5}}{4}, \frac{\sqrt{5}}{2})$, it is easy to check that $bc + a^2 + b = c^2 + ab - 1 = ac - b - b^2 = 0$. Hence, $f_{W^\alpha} = \mathbf{0}$ and so α lifts by Proposition 2.1.

Let $\beta = (012)$. By a similar method, one may prove that $f_{W^\beta} = \mathbf{0}$. (Here, (3.3) and (3.4) are used again and the details are omitted.) Thus, we prove that β also lifts by Proposition 2.1. Since the two generators α and β of A_5 lift, A_5 also lifts. $\square$

3.2. When A/K is isomorphic to $\mathrm{PGL}(2, p)$

In this subsection, we assume that $A/K \cong \mathrm{PGL}(2, p)$ for $p \geqslant 5$ and $n = 1 + p$. Take a fiber F and a vertex $v \in F$. Then, $A_F = A_v K$. Since $(|A : A_F|, |K|) = (1+p, p^3) = 1$ and K is an abelian normal subgroup of A, it follows from [20, I,Hauptsatz 17.4] that K has a complement in A which is isomorphic to $\mathrm{PGL}(2, p)$. We identify K with the corresponding additive group of $V(3, p)$. However, for convenience, we adopt a multiplication notation for K when we consider K as a subgroup of A. For instance, if $k_1 = (x, y, z)$ and $k_2 = (x', y', z')$, then we write $k_1 k_2 = (x + x', y + y, z + z')$. By [1, Lemma 6.3], $\mathrm{GL}(3, p)$ has only one conjugacy class of subgroups isomorphic to $\mathrm{PGL}(2, p)$. Therefore, we may fix a matrix representation of $A/K = \mathrm{PGL}(2, p)$ acting on $K = V(3, p)$ as a subgroup T in $\mathrm{GL}(3, p)$ as follows: First, consider a homomorphism ϕ of $\mathrm{GL}(2, p)$ into $\mathrm{GL}(3, p)$ defined by

$$\phi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (ad - bc)^{-1} \begin{pmatrix} a^2 & 2ab & 2b^2 \\ ac & ad + bc & 2bd \\ c^2/2 & cd & d^2 \end{pmatrix}.$$

It is easy to check that the kernel of ϕ is the center of $\mathrm{GL}(2, p)$, that consists of all scalar multiplications of the identity matrix. Thus, the image T of ϕ is a subgroup in $\mathrm{GL}(3, p)$ which is isomorphic to $\mathrm{PGL}(2, p)$. After given this representation, for any $k = (x, y, z) \in K$ and any matrix $g \in T$, we may write $k^g := (x, y, z)g$.

Due to such a representation, we shall be sloppy and refer to the elements of $\mathrm{PGL}(2, p)$ by 2×2 matrices over $GF(p)$. Take a subgroup $H_1 = \langle t_1 \rangle \rtimes \langle a_1 \rangle \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$ of $\mathrm{PGL}(2, p)$, where

$$t_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad a_1 = \begin{pmatrix} \theta & 0 \\ 0 & 1 \end{pmatrix}$$

for a generator θ of $GF(p)^*$. Let $PG(1, p) = \{\infty, 0, 1, \ldots, p - 1\}$ be the projective line over $GF(p)$, where we identify $\langle(0, 1)\rangle$ and $\langle(1, \ell)\rangle$ with ∞ and ℓ, respectively. Then, H_1 fixes $\infty \in PG(1, p)$ and t_1^i maps ℓ into $\ell + i$. Furthermore, we have $H := \phi(H_1) = \langle t \rangle \rtimes \langle a \rangle$, where $t = \phi(t_1)$ and $a = \phi(a_1)$, and for any i,

$$t^i = \phi(t_1^i) = \begin{pmatrix} 1 & 2i & 2i^2 \\ 0 & 1 & 2i \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad a^i = \phi(a_1^i) = \begin{pmatrix} \theta^i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \theta^{-i} \end{pmatrix}.$$

First, we introduce the following two lemmas.

Lemma 3.3. *Let $M = K \rtimes H$. Then, M has only one conjugate class of subgroups L satisfying $\langle a \rangle \leqslant L \cong H$ and $L \cap K = 1$.*

Proof. Note that $|M| = |K \rtimes H| = |(K \rtimes \langle t \rangle) \rtimes \langle a \rangle| = p^4(p - 1)$. Let $P = K \rtimes \langle t \rangle$. Then, P is a p-group of order p^4. Since $p \geqslant 5$ by assumption, P is a regular p-group (for the definition of regular p-groups, see [20, III, Definition 10.1]). Since $\Phi(P) \leqslant K$ and the order of t is p, P has exponent p. Clearly, M has only one conjugacy class of subgroups isomorphic to $\langle a \rangle$. Assume that L is a subgroup of M such that $\langle a \rangle \leqslant L \cong H$ and $L \cap K = 1$. Then, we may assume that $L = \langle kt\rangle \rtimes \langle a\rangle$ for some $k = (x, y, z) \in K$. Suppose that $(kt)^a = (kt)^i$. Then, we have $(kt)^a = k^a t^a = (\theta x, y, \theta^{-1}z)t^{\theta^{-1}}$ and

$$\begin{aligned}(kt)^i &= (kk^{t^{-1}}k^{t^{-2}}\cdots k^{t^{-i+1}})t^i \\ &= ((x, y, z) + (x, -2x + y, 2x - 2y + z) + \cdots \\ &\quad + (x, -2(i-1)x + y, 2(i-1)^2x - 2(i-1)y + z))t^i \\ &= (ix, -(i-1)ix + iy, \tfrac{(i-1)i(2i-1)}{3}x - (i-1)iy + iz)t^i.\end{aligned}$$

Thus, we get $i = \theta^{-1}$ and

$$(\theta x, y, \theta^{-1}z) = (ix, -(i-1)ix + iy, \frac{(i-1)i(2i-1)}{3}x - (i-1)y + iz).$$

From these two equations, we have $\theta x = ix = \theta^{-1}x$ and so $\theta^2 x = x$. Since $p \geqslant 5$, we get $\theta^2 \neq 1$, and so $x = 0$ and $y = 0$ by the second equation again. Hence, $k = (0, 0, z)$ for any $z \in GF(p)$, that means k has p possibilities. For each k, we get an $L = \langle kt\rangle \rtimes \langle a\rangle$; in particular, $L = H$ when $z = 0$. Furthermore, these p subgroups are conjugate in M. In fact, for any $k = (0, 0, z)$, by taking $k' = (0, \frac{z}{2}, 0)$, we have

$$\begin{aligned}(kt)^{k'} &= k(k')^{-1}tk' = k(k')^{-1}(k')^{t^{-1}}t \\ &= ((0, 0, z) - (0, \tfrac{z}{2}, 0) + (0, \tfrac{z}{2}, -z))t = (0, 0, 0)t = t\end{aligned}$$

and

$$a^{k'} = k'^{-1}ak' = k'^{-1}(k')^{a^{-1}}a = \left((0, -\frac{z}{2}, 0) + (0, \frac{z}{2}, 0)\right)a = a,$$

which forces $L^{k'} = H$, completing the proof. $\square$

Lemma 3.4. *Let $[A : H]$ be the set of right cosets of H in A. Then, in its right multiplication action on $[A : H]$, A has $p - 1$ suborbits of length p not contained in $[M : H]$, which correspond to the $p - 1$ double cosets $Hg(0, y, 0)H$ for any $y \in GF(p)^*$ and $g = \phi(g_1)$, where*

$$g_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Proof. Suppose that the double coset D corresponds to a suborbit of A of length p relative to H not contained in $[M : H]$. Since H has only one conjugacy class of subgroups of order $p - 1$, a must fix a point in this suborbit. Noting that T is 2-transitive on $[T : H]$, we may choose $D = HgkH$ such that $Hgk = Hgka$, in other words, $Hg = Hga^{-1}k^a$, which forces that $Hg = Hga^{-1}$ and $k^a = k$. Hence, we may fix $g = \phi(g_1)$. Assume $k = (x, y, z)$. From $(\theta x, y, \theta^{-1}z) = k^a = k = (x, y, z)$, we have $x = z = 0$ as $\theta \neq \pm 1$, and so $k = (0, y, 0)$, where $y \neq 0$. Therefore, we get $p - 1$ choices for k and so for D. $\square$

Now, $M = K \rtimes H = A_F$ for a fiber F. For any $u \in F$, we have $M_u \cong H$ and $M_u \cap K = 1$. Since M has only one conjugacy class of subgroups isomorphic to $\langle a\rangle$, there exists a vertex $v \in F$ such that $\langle a\rangle \leqslant M_v$. By Lemma 3.3, M_v is conjugate to H in M. It follows that H fixes a vertex in F. Therefore, X is isomorphic to one of $X(A, H, D)$, where $D = Hg(0, y, 0)H$ is as in Lemma 3.4. Moreover, it is easy to see that the $p-1$ graphs corresponding to the $p-1$ choices for D are isomorphic to each other, by changing the basis of $V(3, p)$. Now, we may choose $k = (0, 1, 0)$. Note that

$$g = \phi(g_1) = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}.$$

Since $(gk)^2 = 1$, we get $D = HgkH = H(gk)^{-1}H = D^{-1}$. So, $X(A, H, D)$ is an undirected graph. Clearly, A acts 2-arc-transitively on $X(A, H, D)$, because T is 3-transitive on $V(K_n)$. To complete the proof in this case, we need the following lemma.

Lemma 3.5. *$X(A, H, D) \cong X_3(p)$, and its group of fiber-preserving automorphisms acts 2-arc-transitively.*

Proof. Considering the action of $\mathrm{PGL}(2, p)$ on $PG(1, p)$, one can easily check that for $\ell \in GF(p)^*$ both $g_1 t_1^{\ell} g_1 t_1^{i}$ and $g_1 t_1^{i-\ell^{-1}}$ map ∞ to $i-\ell^{-1}$, respectively. Since $(\mathrm{PGL}(2, p))_\infty = H_1$, we have that for any $i \in GF(p)$, $H_1 g_1 t_1^{\ell} g_1 t_1^{i} = H_1 g_1 t^{i-\ell^{-1}}$ and so under the homomorphism ϕ mentioned before $Hgt^{\ell}gt^{i} = Hgt^{i-\ell^{-1}}$. In addition, $(Hg)gt^i = H$.

By the arguments before the lemma, we know that in the coset graph $X(A, H, D)$, H is adjacent to $Hgkt^{\ell}$ for any $\ell \in GF(p)$. Hence, for any $i \in GF(p)$, Hgt^i is adjacent to $Hgkt^{\ell}gt^{i} = Hgt^{\ell}gt^{i}k^{(t^{\ell}gt^{i})}$ for any $\ell \in GF(p)$. If $\ell = 0$, then

$$Hgt^{\ell}gt^{i}k^{(t^{\ell}gt^{i})} = H(0, 1, 0)gt^{i} = H(0, -1, -2i).$$

Hence, Hgt^i is adjacent to $H(0, -1, -2i)$ for any $i \in GF(p)$, or equivalently, H is adjacent to $Hgt^j(0, 1, 2j)$ for any $j \in GF(p)$. Assume $\ell \in GF(p)^*$ and let $i - \ell^{-1} = j$. Then,

$$\begin{aligned} Hgt^{\ell}gt^{i}k^{(t^{\ell}gt^{i})} &= Hgt^{i-\ell^{-1}}(0, 1, 0)t^{\ell}gt^{i} \\ &= Hgt^{i-\ell^{-1}}(\ell, 2i\ell - 1, 2i^2\ell - 2i) = Hgt^{j}\left(\tfrac{1}{i-j}, \tfrac{i+j}{i-j}, \tfrac{2ij}{i-j}\right). \end{aligned}$$

Hence, Hgt^i is adjacent to $Hgt^j\left(\frac{1}{i-j}, \frac{i+j}{i-j}, \frac{2ij}{i-j}\right)$ for any $i \neq j \in GF(p)$.

Considering the action of $\mathrm{PGL}(2, p)$ on $PG(1, p)$, we may define a bijection from $[\mathrm{PGL}(2, p) : H_1]$ to $PG(1, p)$ by sending H_1 to ∞ and $H_1 g_1 t^i$ to i. Accordingly, we may define a bijection from $[T : H]$ to $PG(1, p)$ by sending H to ∞ and Hgt^i to i. Finally, we may define a map σ from $V(X(A, H, D))$ to $V(X_3(p)) = PG(1, p) \times K$ by sending Hk to (∞, k) and Hgt^ik to (i, k). In viewing the above arguments and the definition of $X_3(p)$, we find that σ is an isomorphism from $X(A, H, D)$ to $X_3(p)$. Moreover, since A acts 2-arc-transitively on $X(A, H, D)$, it follows that for the graph $X_3(p)$, its group of fiber-preserving automorphisms acts 2-arc-transitively. □

The remaining case is to discuss the family $X_3(q)$. It is not convenient to deal with $X_3(q)$ along the proof of Lemma 3.5. Therefore, we give a direct proof by Proposition 2.1.

Lemma 3.6. *For each cover in $X_3(q)$, the group of fiber-preserving automorphisms acts 2-arc-transitively.*

Proof. Recall that $V(K_{1+q})$ is identified with the projective line $PG(1,q) = GF(q) \cup\{\infty\}$. We will adopt the usual computations between ∞ and the elements in $GF(q)$, that is, $\infty + i = \infty$ for $i \in GF(q)$; $\infty i = \infty$ for $i \in GF(q)^*$; and $\frac{\infty}{\infty} = 1$. Let K be the corresponding additive group of $V(3,q)$. Then, $X_3(q) = K_{1+q} \times_f K$ is defined by $f_{i,j} = \left(\frac{1}{i-j}, \frac{i+j}{i-j}, \frac{2ij}{i-j}\right)$ for all $i \neq j$ in $PG(1,q)$.

To prove the lemma, it suffices to show that $\mathrm{PGL}(2,q)$ lifts. For a computation, we identify the element ∞ and any $i \in GF(q)$ in $PG(1,q)$ with $\langle(0,1)\rangle$ and $\langle(1,i)\rangle$ respectively. For a matrix g in $\mathrm{GL}(2,q)$, we denote by $\overline{g}$ the image of g in $\mathrm{PGL}(2,p^\ell)$ under the natural homomorphism. Then, the action of $\overline{g} \in \mathrm{PGL}(2,p^\ell)$ on ∞ and any $i \in PG(1,p^\ell)$ can be written respectively as follows:

$$\infty^{\overline{g}} := \langle(0,1)g\rangle \quad \text{and} \quad i^{\overline{g}} := \langle(1,i)g\rangle.$$

Let

$$g_1 = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad g_4 = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix},$$

where x is a primitive element in $GF(q)$. Then, all of these elements generate $\mathrm{PGL}(2,q)$. In addition, it is easy to check that

$$i^{\overline{g_1}} = ix^{-2}, \quad i^{\overline{g_2}} = i+1, \quad i^{\overline{g_3}} = \frac{i}{i+1}, \quad i^{\overline{g_4}} = ix^{-1},$$

where $i \in PG(1,q)$. In what follows, we show that for $1 \leqslant k \leqslant 4$, $\overline{g_k}$ lifts.

Let W be a closed walk in Y with $f_W = 0$, and for any arc $(i,j) \in A(Y)$, let $\ell_{i,j}$ has the same notation as in Lemma 3.2. Now, we get

$$f_W = \sum_{(i,j)\in A(Y)} \ell_{i,j} f_{i,j} = \sum_{(i,j)\in A(Y)} \ell_{i,j} \left(\frac{1}{i-j}, \frac{i+j}{i-j}, \frac{2ij}{i-j}\right) = \mathbf{0}.$$

Therefore, we have

$$\sum_{(i,j)\in A(Y)} \frac{\ell_{i,j}}{i-j} = 0, \quad \sum_{(i,j)\in A(Y)} \frac{(i+j)\ell_{i,j}}{i-j} = 0, \quad \sum_{(i,j)\in A(Y)} \frac{2ij\ell_{i,j}}{i-j} = 0.$$

Also, we have

$$
\begin{aligned}
f_{W^{\overline{g_1}}} &= \sum_{(i,j)\in A(Y)} \ell_{i,j} f_{i^{\overline{g_1}}, j^{\overline{g_1}}} \\
&= \sum_{(i,j)\in A(Y)} \ell_{i,j} f_{ix^{-2}, ix^{-2}} \\
&= \sum_{(i,j)\in A(Y)} \ell_{i,j} \left(\frac{1}{ix^{-2} - jx^{-2}}, \frac{ix^{-2} + jx^{-2}}{ix^{-2} - jx^{-2}}, \frac{2ix^{-2}jx^{-2}}{ix^{-2} - jx^{-2}} \right) \\
&= \left(x^2 \sum_{(i,j)\in A(Y)} \frac{\ell_{i,j}}{i-j}, \sum_{(i,j)\in A(Y)} \frac{(i+j)\ell_{i,j}}{i-j}, x^{-2} \sum_{(i,j)\in A(Y)} \frac{2ij\ell_{i,j}}{i-j} \right) \\
&= \mathbf{0}.
\end{aligned}
$$

Similarly, we get that $f_{W^{\overline{g_k}}} = \mathbf{0}$, for $k = 2, 3$ and 4. By Proposition 2.1, $\overline{g_k}$ lifts, and so $\mathrm{PGL}(2, q)$ lifts. □

3.3. *When A/K is isomorphic to* $\mathrm{AGL}(3, 2)$

In this subsection, we assume that $A/K \cong \mathrm{AGL}(3, 2) \cong \mathbb{Z}_2^3 \rtimes \mathrm{GL}(3, 2)$, where $K \cong \mathbb{Z}_p^3$ and $p^3 \equiv 1 (\mathrm{mod}\ 7)$. By the arguments in the beginning of Section 3, we have $C_A(K)/K \cong \mathbb{Z}_2^3$. First, we deal with the case that p is an odd prime.

Lemma 3.7. *Let p be an odd prime such that $p^3 \equiv 1 (\mathrm{mod}\ 7)$. Then, there exists no cover occurring.*

Proof. In this case, $|V(K_n)| = 8$. Let F be a fiber and let $v \in F$. Then, $A_F = A_v K$. Since $(|A : A_F|, |K|) = (8, p^3) = 1$ and K is an abelian normal subgroup of A, by [20, I, Hauptsatz 17.4], K has a complement in A, say G. Thus, $A = KG$, where $G \cong \mathrm{AGL}(3, 2) \cong \mathbb{Z}_2^3 \rtimes \mathrm{GL}(3, 2)$. Let $G = L \rtimes T$, where $L \cong \mathbb{Z}_2^3$ and $T \cong \mathrm{GL}(3, 2) \cong \mathrm{PSL}(2, 7)$. Then, $C_A(K) = K \times L$ and $A = KLT$. Since KT is the stabilizer of a fiber and KT has only one conjugacy class of subgroups isomorphic to T, we get $T = A_u$ for some u in the fiber. So, X is isomorphic to a coset graph $X(A, T, D)$, where $D = T\ell kT$ for some $\ell \in L$ and $k \in K \setminus \{0\}$. Thus, D corresponds to a suborbit of A of length 7 relative to T, or equivalently, T has an orbit of length 7 in its conjugacy action on K. In other words, as a subgroup of $\mathrm{GL}(3, p)$, T has an orbit of length 7 in its action on $V(3, p)$. However, it is not true by Lemma 2.7. Hence, there exist no overs occurring in this case. □

To complete the proof of the main theorem, we now consider the case of $p = 2$ and $K \cong \mathbb{Z}_2^3$. Let $C = C_A(K)$. Then, C acts regularly on $V(X)$ and $C/K \cong \mathbb{Z}_2^3$. Now, C is an extension of K by $\mathbb{Z}_2^3$ and so it has the exponent either 2 or 4. Let $T = A_v$ for some $v \in V(X)$. Then, $T \cong \mathrm{GL}(3, 2) \cong \mathrm{PSL}(2, 7)$ and $A = C \rtimes T$. Since C/K is elementary abelian, we get $\Phi(C) \leqslant K$. Since T normalizes C, it normalizes $\Phi(C)$. On the other hand, since T acts on K nontrivially and T is simple, K is a minimal normal subgroup in A. It follows that $\Phi(C)$ is trivial or K. Thus, C is isomorphic to either $\mathbb{Z}_2^6$ or a 2-group generated by three elements of order 4. Suppose that the latter case happens. Then, $\Phi(C) = Z(C) = K$. A direct checking from a classification of groups of order 2^6 (see [19]) shows that C cannot be nonabelian. Therefore, C must be the group G in Lemma 2.8. However, the conjugacy action of T on C gives an imbedding of T in $\mathrm{Aut}\, C$, and in our case, T has an orbit of length 7 on C, contradicting to Lemma 2.8. Hence, it must be $C \cong \mathbb{Z}_2^6$. Before treating this case, we first prove a more general result (see Lemma 3.8).

Let Ω be a finite set and let $V = V(\Omega)$ be the set of all characteristic functions χ_Δ for $\Delta \in P(\Omega)$, which are functions from Ω to $GF(2)$. As stated in Example 1.2(4), V forms a vector space over $GF(2)$ with the natural basis $\{\chi_{\{i\}} \mid i \in \Omega\}$. Assume that G is a permutation group on Ω. Then, V can be defined as a G-module, called a *permutation module*, where the action of $g \in G$ is defined by $(\chi^g)(i) = \chi(i^{g^{-1}})$ for all $i \in \Omega$, see [27]. Moreover, the action of G on V preserves a natural inner product on V, defined by $(\chi, \chi') = \sum_{i\in\Omega} \chi(i)\chi'(i)$. Clearly, for each G-submodule M of V, the orthogonal complement $M^\perp$ of M is also a G-submodule. The group G clearly leaves invariant the two spaces I and $I^\perp$, where I consists of the constant functions and its orthogonal $I^\perp$ consists of all functions χ satisfying $\sum_{i\in\Omega} \chi(i) = 0$, where the dimensions of I and $I^\perp$ are, respectively, 1 and $|\Omega| - 1$.

Lemma 3.8. *For the cover $X_4(m)$ in Example* 1.2(4), *the group of fiber-preserving automorphisms acts* 2*-arc-transitively.*

Proof. For $m \geqslant 3$, let $\Omega = PG(m-1, 2)$, the $(m-1)$-dimensional projective space over $GF(2)$, while we identify Ω with $V(m, 2) \setminus \{0\}$. Let $G = \mathrm{PGL}(m, 2) \cong \mathrm{PSL}(m, 2) \cong \mathrm{GL}(m, 2)$. Then, the $(2^m - 1)$-dimensional space $V(\Omega)$ can be viewed as a permutation G-module via the natural action of G on Ω. For $i = 0, \ldots, m-1$, let V_i be the subspace of $V(\Omega)$ generated by the characteristic functions of all i-dimensional subspaces of $PG(m-1, 2)$. Then, $V_0 = V(\Omega)$, $V_{m-1} = I$, and V_i is a G-submodule. Choose a basis $\{\alpha_1, \ldots, \alpha_m\}$ for $V(m, 2)$. Then, it is easy to check that $\{\chi_{\{\alpha_i\}} + V_1 \mid 1 \leqslant i \leqslant m\}$ (resp. $\{\chi_{\{\alpha_i,\alpha_j,\alpha_i+\alpha_j\}} + V_2 \mid i \neq j, 1 \leqslant i, j \leqslant m\}$) is a basis for the irreducible quotient G-module V_0/V_1 (resp. V_1/V_2). Therefore, the G-module V_0/V_2 of dimension $(m + \frac{m(m-1)}{2})$ has the irreducible G-submodule V_1/V_2. Consider the affine transformation group $\mathrm{AGL}(m + \frac{m(m-1)}{2}, 2)$ of the linear vector space V_0/V_2, with the translation (characteristic) subgroup, say H. Then, G can be viewed as a subgroup in $\mathrm{AGL}(m + \frac{m(m-1)}{2}, 2)$. Let $A^* = H \rtimes G$, which is transitive on V_0/V_2 with the point-stabilizer G. Fix a point x in Ω and let h_x be the translation corresponding to $\chi_{\{x\}} + V_2$. Then, we define a coset graph $X^* =: X(A^*, G, Gh_xG)$. Since the group G is 2-transitive on Ω, A^* acts 2-arc-transitively on the graph X^*. Let K be a subgroup of H which is generated by the translations of $\chi_{\{i,j,i+j\}} + V_2$ for all $i \neq j$ in Ω. Then, K is normal in A^* and it induces the blocks $\{Ghk \mid k \in K\}$ for any $h \in H$. Then, the quotient (block) graph of X^* induced by K is isomorphic to the complete graph K_{2^m} and consequently, X^* is a $2^{\frac{m(m-1)}{2}}$-fold regular cover of K_{2^m}, whose fiber-preserving automorphism subgroup A^* acts 2-arc-transitively. The remaining argument is to show that $X^* \cong X_4(m)$.

We first define a bijection ϕ from $V(X^*)$ to V_0/V_2 by sending Gh to $\chi_\Delta+V_2$, where $h \in H$ is the translation corresponding to $\chi_\Delta + V_2$ for $\Delta \subset V(m, 2)$. With the map ϕ, the blocks (fibers) induced by K corresponds to $\{\chi_{\{i\}} + V_2, \quad \chi_{\{i\}} + \chi_{\{j,\ell,j+\ell\}} + V_2 \mid j \neq \ell \in \Omega\}$ for any $i \in V(m, 2)$. Then, we define another bijection σ from V_0/V_2 to $V(X_4(m)) = V(m, 2) \times (V_1/V_2)$ by sending $\chi_{\{i\}} + \chi_{\{j,\ell,j+\ell\}} + V_2$ to $(i, \chi_{\{j,\ell,j+\ell\}} + V_2)$. Therefore, $\sigma\phi$ is a bijection from $V(X^*)$ to $V(X_4(m))$. For $i \neq j$ in Ω and an line L contained in Ω such that $j \notin L$, let h_i and $h_{\{j\}\cup L}$ be the translations corresponding to $\chi_{\{i\}} + V_2$ and $\chi_{\{j\}\cup L} + V_2$, respectively. Then, Gh_i and $Gh_{\{j\}\cup L}$ are adjacent in X^* if and only if $h_{\{j\}\cup L}h_i^{-1} \in Gh_xG$, or equivalently, $h_{\{j\}\cup L} = g^{-1}h_xgh_i$ for some $g \in G$ (noting the structure of A^*). Therefore, we have $\chi_{\{j\}} + \chi_L + V_2 = \chi_{\{x^g\}} + \chi_{\{i\}} + V_2$, which implies $\chi_L + V_2 = \chi_{\{x^g\}} + \chi_{\{i\}} + \chi_{\{j\}} + V_2$. It gives $\chi_L + V_2 = \chi_{\{i,j,i+j\}} + V_2$. Hence, $\sigma\phi$ induces an isomorphism from the coset graph X^* to the cover $X_4(m)$. In particular, for $X_4(m)$, the group of fiber-preserving automorphisms acts 2-arc-transitively. □

Continuing the arguments before Lemma 3.8, we finish the proof of Theorem 1.3 with the following lemma.

Lemma 3.9. *If $C \cong \mathbb{Z}_2^6$, then $X \cong X_4(3)$ and A acts 2-arc-transitively on X.*

Proof. In our case, C is regular on $V(X)$ and so $V(X)$ can be identified with C, and both of them can be identified with $V(6, 2)$. Let $T = A_0$, where $0 \in V(X) = V(6, 2)$. Then, T is 2-transitive on the neighborhood $\Omega := X_1(0)$ of 0 with size 7 and Ω is a generating subset of C, because X is connected. Since $A = C \rtimes T$, C is a T-module in the conjugacy action, and K is a 3-dimensional T-submodule. For Ω, let $V = V(\Omega)$ be the corresponding permutation T-module. Then, we have two T-modules, that is the 6-dimensional module C and the 7-dimensional module V. Clearly, the map ϕ: $V \to C$ defined by $\sum_{i\in\Omega} k_i\chi_{\{i\}} \mapsto \sum_{i\in\Omega} k_i i$ is a T-module epimorphism. It is easy to see that $I \subset \operatorname{Ker} \phi$ and so $\overline{V} := V/I \cong C$ is a faithful T-module. Therefore, we may identify $V(X)$ with $\overline{V}$ and then all the arguments follow from Lemma 3.8. Note that in particular, in the case $m = 3$, we have that $V_2 = I$ and $V_1 = V_2 \oplus W$, where W is the 3-dimensional subspace generated by all the characteristic functions of the complements of lines in $PG(2, 2)$. By [21, Theorem 5.1], W is the unique faithful minimal T-submodule of V as well as $\overline{V}$. □

As the final of this paper, we introduce the following proposition.

Proposition 3.10. *All of the covers contained in $X_1(p)$, $X_2(p)$, $X_3(p)$ and $X_4(3)$ are minimal.*

Proof. To prove the proposition, it suffices to show that each of these graphs X cannot be a cover of a graph Z, which also is a cover of the complete graph K_n whose covering transformation group K_0 is isomorphic to either $\mathbb{Z}_p$ or $\mathbb{Z}_p^2$. By [5], we have that either $K_0 = \mathbb{Z}_2$ and $n \geqslant 4$; or $K_0 = \mathbb{Z}_2^2$ and $n = 1 + q$, where q is a prime power such that

$q \equiv 1 (\mathrm{mod}\ 4)$. Therefore, noting the values of n and p in the graphs $X_i(p)$ for $i = 1,2$ and 3, and $X_4(3)$, we only need to consider the case $X_1(2)$, where $n = 4$, $K_0 = \mathbb{Z}_2$, and actually, Z is now the cube and $X_1(2)$ is a 4-fold cover of Z. In what follows, we show that this cannot happen.

Suppose that $X_1(2)$ is a 4-fold cover of the cube Z with the covering transformation group $K_1 \cong \mathbb{Z}_2^2$. Let A and K have the meaning as before. Then, K_1 is normal in A and $K/K_1 \cong \mathbb{Z}_2$. Note that $A/K \cong S_4$ and $A/K_1 \cong \mathrm{Aut}\, Z \cong S_4 \times \mathbb{Z}_2$, we have that $C_A(K_1)$ contains some Sylow 2-subgroups of A which is transitive on $V(X_1(2))$. Therefore, all the elements of even order of $\mathrm{Aut}\, Z$ preserve the voltages of any closed walk in the base graph Z by Proposition 2.2. With these arguments, by considering the action of a Sylow-2 subgroup of $\mathrm{Aut}\, Z$ on the cycles of length 4 in the cube Z, we can easily see that if we choose a spanning tree for Z such that the voltages on the tree-arcs are assumed to be trivial, then all the voltages on the cotree-arcs would be the same, which forces $X_1(2)$ to be disconnected, a contradiction. □

Acknowledgments

Partially supported by the NSFC(19901022, 19472024), BNSF(19920003), SYSF (19981002), SRF for ROCS, SEM of China and Com^{2}MaC-KOSEF. The first and the third authors would like to thank the Com^{2}MaC at POSTECH, Korea for its hospitality. Most of the work was done when they visited there. Also, the authors are indebted to the referees who gave us many helpful suggestions.

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